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BEHAVIOUR OF A BOSE-EINSTEIN CONDENSATE NEAR THE ZERO DISPERSION POINT*

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Abstract
A cigar shaped BEC in a periodic external field is analyzed using the multiple scales method. Usually the dominant amplitude satisfies the completely integrable NLS equation. The discussion is extended to the vicinity of the "zero-dispersion point" (the point where the coefficient of the second order derivative vanishes). The multiple scales method is adapted to this situation and an equation containing the third order derivative is found for the dominant amplitude. It is no more integrable and several properties of it are investigated.
Keywords: Bose-Einstein condensate, multiple scales method, zero dispersion.

1. Introduction

Since the first observation of Bose-Einstein condensation (BEC) in vapors of rubidium and other alkali gases in 1995 (for a brief history see [1], [2]), the field enjoyed remarkable developments, both from experimental and theoretical point of view. A very promising achievement was the use of magnetic-field Feschbach resonance for manipulating degenerate atomic gases [3]. With this technique the scattering length is tuned from positive to negative values, and many interesting experimental possibilities are opened [4]. Among them the creation of different kinds of solitons in BEC in a cigar shaped geometry has received a great deal of attention (see [5], [6] and references therein).

A special situation arises when a periodic potential (generated by detuned standing laser waves, an optical lattice) is superposed on the condensate. Then a fragmentation of the original wave function and a crystal-like structure of mutually interacting BECs appears,

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opening the possibility to observe several macroscopic interesting phenomena. Studies in this direction were done by several authors [7]-[10].

In the present paper a BEC in a cigar shaped geometry is considered, with a periodic potential along x-axis. As is well known, for a low density condensate the wave function of the system satisfies the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\varphi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\vec{r})+g|\varphi|^2\right)\varphi \tag{1}$$

where $g=\frac{4\pi\hbar^2 a_s}{m}$, with a_s the s-wave scattering length of atoms (it can be either positive or negative). The trap potential is assumed of the form

$$V(\vec{r})=\frac{1}{2}m\omega_\perp^2 r_\perp^2+V_0V(x)$$

Here $\omega_\perp$ is the oscillating frequency in the radial direction. Introducing the dimensionless quantities

$$t\to\frac{2t}{\omega_\perp},\qquad \vec{r}\to a_\perp\vec{r},\qquad \psi=\sqrt{\frac{N}{a_\perp^3}}\varphi$$

(N is the total number of atoms in the condensate), equation (1) becomes

$$i\frac{\partial\psi}{\partial t}=\left(L+\chi|\psi|^2\right)\psi \tag{2}$$

where $\quad L=L_\perp+L_x,\qquad L_\perp=-\nabla_\perp^2+r_\perp^2,\qquad L_x=-\frac{\partial^2}{\partial x^2}+\Lambda V(x)$

with $\Lambda=\frac{V_0\omega_\perp}{\hbar}$ and $\chi=\frac{8\pi N a_s}{a_\perp}$, and the wave function $\psi(\vec{r})$ is normalized to unity, $\int d\vec{r}_\perp\int dx|\psi(\vec{r})|^2=1$ and satisfies the periodicity condition $\psi(x+l)=\psi(x)$ (l the dimensionless length of the cylinder).

2. Linearized problem

The solution of the linear problem can be written as a product of a harmonic oscillator wave function (in the fundamental state) and a Bloch function

$$\psi_{0,\nu q}(\vec{r},t)=\frac{1}{\pi}e^{-\frac{1}{2}\vec{r}_\perp}\frac{1}{\sqrt{M}}e^{iqx}u_{\nu q}(x)e^{-iE_{0,\nu}(q)t}=\varphi_0(\vec{r}_\perp)\varphi_{\nu q}(x)e^{iE_{0,\nu}(q)t} \tag{3}$$

corresponding to the energy

$$E_{0,\nu}(q)=\hbar\omega_\perp+E_\nu(q).$$

We can assume small nonlinearity ($\chi|\psi|^2 << 1$) and therefore we can restrict ourselves to the lowest state of the harmonic oscillator in the radial variable. In (3) $M = l/a$ (a the period of $V(x)$), q is the wave vector restricted to the first Brillouin zone (-π/*a,* π/*a,*), $E_\nu(q)$ the energy of the corresponding Bloch function, and $u_{\nu q}(x)$ is a periodic function $u_{\nu q}(x+a) = u_{\nu q}(x)$ normalized to unity in the unit cell.

3. Multiple scales analysis

The solution (3) is unstable to small modulations of the amplitude. The usual way to treat this instability is to use the asymptotic method of multiple scales [11], [12]. Besides the "fast variables" one introduces the "slow variables"

$$x_n = \varepsilon^n x\,, \qquad t_n = \varepsilon^n t \tag{4}$$

and the wave function ψ is expanded in the small parameter ε

$$\psi = \sum_{j=1}^{\infty} \varepsilon^j \psi_j \tag{5}$$

where each component ψ_j is a function both of fast variables $(\bar{r}_\perp, x, t)$ and the slow variables $\{\bar{x} \equiv (x_1, ...); \bar{t} \equiv (t_1, ...)\}$. In the first order in ε we get

$$i\frac{\partial \psi_1}{\partial t} - L\psi_1 = 0 \tag{6}$$

and consequently

$$\psi_1 = A(\bar{x}, \bar{t})\psi_{0,\nu q}(\vec{r}, t) \tag{7}$$

where the amplitude *A* depends only on the slow variables. In order ε^2 we obtain

$$i\frac{\partial \psi_2}{\partial t} - L\psi_2 = -i\frac{\partial \psi_1}{\partial t_1} - 2\frac{\partial^2 \psi_1}{\partial x \partial x_1} \tag{8}$$

The right-hand side of (8) contains terms proportional with $\psi_{0,\nu q}(\vec{r}, t)$ and consequently secular terms (proportional with t) in ψ_2 can appear. To avoid such situations one has to assume for ψ_2 an expansion

$$\psi_2(\vec{r}, t; \overline{x}, \overline{t}) = \sum_{\nu'} B_{\nu' q}(\bar{x}, \bar{t})\varphi_0(\vec{r}_\perp)\varphi_{\nu q}(x)e^{-iE_{0,\nu'}(q)t} \tag{9}$$

where the new amplitudes $B_{\nu' q}$ are functions only on the slow variables $(\bar{x},\bar{t})$. It is necessary to leave the amplitude $B_{\nu q}$ undetermined, and it is easily seen that this can be realized if $A(\bar{x},\bar{t})$ satisfies the equation

$$\frac{\partial A}{\partial t_1}+c(q)\frac{\partial A}{\partial x_1}=0 \tag{10}$$

where

$$c(q)=2\left[q-i\int_0^a u_{\nu q}^*(x)\frac{d}{dx}u_{\nu q}(x)dx\right] \tag{11}$$

The relation (10) is satisfied if A depends on (x_1,t_1) only through the combination

$$\xi=x_1-c(q)t_1. \tag{12}$$

The other amplitudes $B_{\nu' q}$ ($\nu'\neq 0$) are now easily determined, namely

$$B_{\nu' q}=\frac{\partial A}{\partial \xi}\frac{\Gamma_{\nu'\nu}(q)}{E_{\nu'}(q)-E_{\nu}(q)} \tag{13}$$

$$\Gamma_{\nu'\nu}(q)=2\int_0^a u_{\nu' q}^*(x)\frac{d}{dx}u_{\nu q}(x)dx.$$

In the third order in ε appears also a nonlinear contribution. We get

$$i\frac{\partial \psi_3}{\partial t}-L\psi_3=-i\frac{\partial \psi_2}{\partial t_1}-2\frac{\partial^2\psi_2}{\partial x\partial x_1}-i\frac{\partial \psi_1}{\partial t_2}-\left[\frac{\partial^2}{\partial x_1^2}+2\frac{\partial^2}{\partial x\partial x_2}\right]\psi_1+\chi|\psi_1|^2\psi \tag{14}$$

Also we are faced with the posibility to have a secular behaviour. To avoid it we have to assume that $B_{\nu q}$ (and consequently all the amplitudes $B_{\nu' q}$) are depending on (x_1,t_1) only through the variable ξ, and we have to impose a certain restriction on A. Denoting

$$\bar{\chi}=\chi\int d\vec{r}_\perp\varphi_0^4(\vec{r}_\perp)\frac{1}{M}\int_0^l dx|u_{\nu q}(x)|^4$$

$$D_2=1+\sum_{\nu'\neq\nu}\frac{\Gamma_{\nu\nu'}(q)\Gamma_{\nu'\nu}(q)}{E_{\nu'}(q)-E_{\nu}(q)} \tag{15}$$

after straightforward calculation this restriction writes

$$i\frac{\partial A}{\partial t_2}+D_2\frac{\partial A}{\partial \xi}-\bar{\chi}|A|^2A=0 \tag{16}$$

(only one low spatial variable, x_1, is taken into account). The equation (16) is the well known nonlinear Schrödinger equation, and depending on the signs of D_2 and $\bar{\chi}$ it has bright or dark soliton solutions. The coefficient D_2, defined in (15) is related to the second order dispersion.

Usually there is a point $q = q_c$ for which $D_2(q_c) = 0$, which we shall call the "zero dispersion point" and will be discussed in the next section.

4. Zero dispersion point

The zero dispersion point problem is well known in nonlinear optical fibers, being one of the most convenient way of operating [13]-[16]. The same situation was discussed in nonlinear quasi-one-dimensional molecular crystals (Davydov's model) [17].

To discuss the ZDP case we have to modify the multiple scales method. Instead of (5) we have to use the following expansion

$$\psi = \sqrt{\varepsilon}\sum_{j=1}\varepsilon^j \psi_j \tag{17}$$

and to consider the wave vector q in the neighborhood of q_c, namely $q = q_c + \varepsilon\Delta q$. The first two orders in ε give the same results as in the previous section. In order $\varepsilon^{7/2}$ we get

$$i\frac{\partial\psi_3}{\partial t} - L\psi_3 = -i\frac{\partial\psi_2}{\partial t_1} - 2\frac{\partial^2\psi_2}{\partial x\partial x_1} - i\frac{\partial\psi_1}{\partial t_2} - \left(\frac{\partial^2}{\partial x_1^2} + 2\frac{\partial^2}{\partial x\partial x_2}\right)\psi_1$$

In order to eliminate possible secular behavior the right-hand side of this equation has to be orthogonal on $\varphi_{0,\nu q}(x,t)$. One obtains

$$-i\left(\frac{\partial A}{\partial t_2} + c\frac{\partial^2 A}{\partial x\partial x_2}\right) - i\left(\frac{\partial B}{\partial t_1} + c\frac{\partial^2 B}{\partial x\partial x_1}\right) - D_2(q_c + \varepsilon\Delta q)\frac{\partial^2 A}{\partial x_1^2} = 0$$

which is satisfied if A depends on (t_2, x_2) through the combination $\eta = x_2 - ct_2$ and B depends on (t_1, x_1) through $\xi = x_1 - ct_1$. The last term gives a contribution in the next order, $\varepsilon^{9/2}\left(\frac{dD_2}{dq}\right)_{q=q_c}\Delta q$. We can look now for the following expression of ψ_3

$$\psi_3 = \sum_{\nu'} C_{\nu\nu'}(\bar{x},\bar{t})\varphi_{0,\nu q}(x,t) \tag{18}$$

and after straightforward calculation we get $(\nu' \neq \nu)$

$$C_{\nu\nu'} = \frac{\Gamma_{\nu\nu'}}{E_{\nu'} - E_\nu}\left(\frac{\partial A}{\partial\eta} + \frac{\partial B}{\partial\xi}\right) + \frac{\Gamma_{\nu\nu'}(\Gamma_{\nu'\nu'} - \Gamma_{\nu\nu})}{(E_{\nu'} - E_\nu)^2}\frac{\partial^2 A}{\partial\xi^2} \tag{19}$$

while the coefficient $C_{\nu\nu}$ remains undetermined.

The relevant order is $\varepsilon^{9/2}$. One obtains

$$i\frac{\partial\psi_4}{\partial t}-L\psi_4=-i\frac{\partial\psi_3}{\partial t_1}-2\frac{\partial^2\psi_3}{\partial x\partial x_1}-i\frac{\partial\psi_2}{\partial t_2}-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x\partial x_2}\right)\psi_2-$$
$$-i\frac{\partial\psi_1}{\partial t_3}-2\left(\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial x\partial x_3}\right)\psi_1+\overline{\chi}|\psi_1|^2\psi$$

In order to eliminate a possible secular behavior again we have to ask that the right-hand side to be orthogonal to the null space of the left-hand side. After straightforward calculations and using the previous expressions for ψ_3 and ψ_2 we finally get

$$i\left(\frac{\partial A}{\partial t_3}+c\frac{\partial A}{\partial x_3}\right)+iM\frac{\partial^3}{\partial\xi^3}+N\frac{\partial^2 A}{\partial\xi^2}-\overline{\chi}|A|^2A=0 \tag{20}$$

where

$$M=-i\left(\sum_{\nu'\neq\nu}\frac{\Gamma_{\nu\nu'}\Gamma_{\nu'\nu}\left(\Gamma_{\nu'\nu'}-\Gamma_{\nu\nu}\right)}{\left(E_{\nu'}-E_\nu\right)^2}\right)_{q=q_c}$$

$$N=\left(\frac{d}{dq}\sum_{\nu'\neq\nu}\frac{\Gamma_{\nu\nu'}\Gamma_{\nu'\nu}}{\left(E_{\nu'}-E_\nu\right)}\right)_{q-q_c} \tag{21}$$

Let us assume that A is independent of x_3. Also the dependence on η is irrelevant at this stage. We can eliminate the second derivative in (21) using the transformation

$$X=\xi+\alpha t_3 \qquad T=t_3$$

$$A(\xi,t_3)=\psi(X.T)e^{i\varphi} \tag{22}$$

$$\varphi(\xi,t_3)=\beta(\xi-\gamma t_3)$$

$$\alpha=\frac{N^3}{3M,} \qquad \beta=\frac{N}{3M}, \qquad \gamma=\frac{N^2}{3M}-\frac{3M^2}{N}.$$

One obtains

$$i\frac{\partial\psi}{\partial T}+iM\frac{\partial^3\psi}{\partial X^3}-\overline{\chi}|\psi|^2\psi=0 \tag{23}$$

which is no more a completely integrable equation.

5. Conclusions

A Bose-Einstein condensate in a cigar shape geometry is studied when a periodic potential in the longitudinal direction is present. The condensate is assumed to remain in the ground state of the radial harmonic oscillator. The longitudinal Bloch wave function is unstable at small modulations of the amplitude. The multiple scales method is used to discuss this phenomenon. When the condensate is far from the so called zero dispersion point the main amplitude satisfies the well known nonlinear Schrödinger equation. The zero dispersion point separates the region where bright solitons exist from the region where only dark ones appear. In the vicinity of this point the multiple scales method has to be slightly modified and the main amplitude satisfies now a non-integrable equation.

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