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RICCI FLOW EQUATION IN TWO DIMENSIONS AND THE LINEARIZATION[1] APPROACH

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Abstract
Ricci flow equation in two dimensions is investigated in the conformal gauge in a linearization approach. Using a non-linear substitution of logarithmic type, the emergent quadratic equation is split in various ways. New special solutions involving arbitrary functions are presented. Some special reductions are also discussed.
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1. Introduction

As an attempt to quantize gravity, it is interesting to investigate quantum field theory in a curved space-time background. Solving relativistic field equations in (3+1) -dimensional curved space-time is generally a difficult process. An alternative approach is to consider lower-dimensional space-times models where exact solutions may be obtained. It is now long time since lower-dimensional gravity proved to exhibit many of the qualitative features of (3+1) -dimensional general relativity, high dimensional black holes, cosmological models and branes [1,2,3].

It is a long time since 1+1 and 0+1 dimensional gravity coupled to scalar fields proved to be a reliable model for high dimensional black holes, cosmological models and branes. The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory: symmetry reduction, compactification, holographic principle, AdS/CFT correspondence, duality, etc. [4].

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Ricci flow is an important geometric evolution equation in Riemannian geometry. It was introduced by R. Hamilton in 1982 [H] for producing Einstein metrics of positive scalar curvature and constant positive sectional curvature. In two space-time dimensions, Ricci flow provides a proof of the uniformization theorem, which states that every closed orientable two-dimensional manifold with handle number 0 , 1 or >1 admits uniquely the constant curvature geometry with positive, zero or negative curvature, respectively. On the other hand and it was used extensively to try to prove some outstanding results on 3 - and 4 -dimensional manifolds like the Thurston'd geometrization conjecture, classification of compact 4 -manifolds with non-negative isotropic curvature, etc. [6,7].

The Ricci flow equations arose independently in physics in the early '80s. Since then they have become a major tool for addressing a variety of problems in the quantum theory of fields and strings as well as in geometry where some ground breaking results have been obtained in recent years. The Ricci flow of d -dimensional manifolds is interesting because of its relationship to the renormalization group equations of generalized sigma models with d-dimensional target space. On the other hand it is well known the connection between the sigma models from physics and the harmonic maps in mathematics.

The two dimensional Ricci flow equation arises in many areas of physics where super fast diffusion processes take places. It appears in studies in a certain approximation to Carleman's model of the Boltzmann equation. It is used to describe some diffusion processes in plasma, including mirror effects. It governs the expansion of a thermalized electron cloud described by isothermal Maxwell distribution. Finally, it appears as limiting case of the porous medium equation. On the other hand the famous cigar soliton coincides with the so called Baremblatt solution in the theory of porous media, whereas the sausage deformation of the round sphere coincides with the axi-symmetric solution found in fast and super fast diffusion processes when written in conformally flat frame.

In two dimensions, the Ricci flow equation written in a local system of conformally flat coordinates has been studied in detail by Bakas [8,9,10,11] from an algebraic point of view. It was considered as a continual version of the general Toda-type equation for a given Lie algebra. It was found a formal power series solution by expanding the path-ordered exponentials. Although the proposed general solution provides a formal complete solution for the Ricci flow equation, its form is quite intricate and difficult to handle. In a recent paper [CV] we used a direct non-linear substitution, then split the resulting non-linear equation. We considered the Ricci flow equation to be in the class of linearizable systems rather than Lax-pair solvable ones. This results in a class of special solutions, which recover practically all known solution in the literature. Therefore the linearization approach represents a powerful procedure to generate explicit solutions of the Ricci flow equation. Adding supplementary assumptions connected with the symmetries of the concrete physical systems or assuming special dependence on parameter of deformation it is possible to get effectively large class of solutions.

The Ricci flows are second order non-linear parabolic differential equations for the components of the metric $g_{\mu\nu}$ of an n -dimensional Riemannian manifold which are driven by the Ricci curvature tensor $R_{\upsilon\nu}$:

$$\frac{\partial}{\partial t} g_{\mu\nu} = -R_{\mu\nu} \tag{1}$$

This equation describes geometric deformations of the metric $g_{\mu\nu}$ with parameter *t*. From the analysis of such partial differential equation one obtains existence and uniqueness theorems of solutions on some interval of the parameter of deformations starting from any smooth initial metric. In some cases the solutions exist after infinitely long run of the parameter of deformation in the sense that the metric does not become singular anywhere.

The connection between high and low dimensions has been demonstrated in different context of gravity and string theory and in some cases allowed to find general solutions or some special classes of solutions in high dimensional theories. Realistic theories describing black holes and cosmologies are usually non integrable. However, explicit general solutions of the integrable approximations in lower dimensions may allow the construction of different sorts of perturbation theories. Therefore looking for new exact, analytic solutions of the Ricci flow equations in lower dimensions it is possible to find solutions of interest for a realistic description of gravitational interaction in higher dimensions.The Ricci flow equations on 2- and 3-dimensional manifolds have attracted considerable attention in physical literature in connection with lower dimensional black hole geometry, exact solutions of the renormalization group equations in quantum field theories, description of the decay of singularities in non-compact spaces, etc.

The purpose of this paper is to present new explicit solutions for 2 -dimensional Ricci flow equation using a linearization approach.

2. Linearization approach

In what follows we shall consider in a 2 -dimensional space a local system of conformally flat coordinates in which the metric has the form

$$ds_t^2 = \frac{1}{2} e^{\Phi(x,y;t)} (dx^2 + dy^2) = 2e^{\Phi(z+,z;t)} dz_+ dz_-$$

using Cartesian coordinates x,y or the complex conjugate variables $2z_\pm = y \pm ix$.

Having in mind that the only non-vanishing component of the Ricci tensor is

$$R_{+-} = -\partial_+ \partial_- \Phi(z_+, z_-; t)$$

the Ricci flow equation (R) becomes

$$\frac{\partial}{\partial t} e^{\Phi(z+,z\_;t)} = \partial_+ \partial_- \Phi(z_+, z_-; t). \quad (2)$$

It is useful to use the substitution

$$v(z_+, z_-; t) = e^{\Phi(z+,z\_;t)}$$

writing eq. (2) in the form

$$(v)_t = (\ln v)_{z+z\_} \quad (3)$$

This equation has been studied in detail from the algebraic point of view in [10]. It is considered as a kind of a general Toda equation with a "continuous" Cartan matrix in a form of derivative of Dirac distribution in the nonlinear part. The Lax pair formulation of the initial Toda equation is "translated" in the continual version together with its general power series solution. In spite of its elegance, in this approach everything goes only formally. The corresponding Lie algebra is rather an exotic infinite dimensional one and, accordingly, the integrability through zero curvature formulation is problematic as well [10]. In any case, starting from a simple "seed" solution the power series formula can be effectively implemented to give approximate solutions. Our approach to the equation is somewhat different. We use a direct nonlinear substitution, then split the resulting nonlinear equation. This results in a class of special solutions. Moreover we consider that the equation (3) is rather in the class of *linearizable* systems than Lax-pair solvable ones.

Our supposition comes from the following observations:

1. Starting with (3) and using the substitution $v = \varphi_{z+}$, after integrating once with respect to z_+ we get

$$\varphi_t \varphi_{z+} = \varphi_{z+z-} + C\varphi_{z+} \quad (4)$$

where C should be a function of z_- and t , but for the moment it is considered constant. Now making the substitution $\varphi = \ln F$ we will end up with the following quadratic equation:

$$F_t F_{z+} - F_{z+z-} F + F_{z+} F_{z-} - CFF_{z+} = 0 \quad (5)$$

2. Equation (3) does *not* pass the Painlev e test.

Accordingly we are going to seek an underlying linear (or solvable) system for (3). Let us remark that eq. (3) is symmetric in variables z_+ and z_- and, consequently, in the rest of the paper, the role of these variables can be interchanged.

Let us assume that C is no longer a constant, but a free function of z_- and t. In this case, defining

$$\varphi = \psi + \int_{-\infty}^{t} C(z_-, t)dt$$

then (4) will have the form

$$\psi_{z+}\psi_t = \psi_{z+z-} \tag{6}$$

Using the same nonlinear substitution $\psi = \ln F$ we get

$$F_{z+}(F_t + F_z) = FF_{z+z-} \tag{7}$$

Equation (7) can be split in some linear or nonlinear solvable equations. Of course, all the possibilities we are going to analyze will give only special solutions and not general ones. First of all, we shall split eq. (7) into a system of linear equations. Here we list the most interesting possibilities:

- *Linearization I*

$$\begin{aligned} F_{z+z-} &= 0 \\ F_t + F_z &= 0 \end{aligned}$$

with the general solution

$$F(z_+, z_-,;t) = f(z_+) + g(t - z_-)$$

where *f,g* are *arbitrary functions*. The solution of (2) is

$$v(z_+, z_-;t) = \frac{f(z_+)}{f(z_+) + g(t - z_-)} \tag{8}$$

which is in fact the generalization of the multi-shock solution.

Other linearizations of this type are presented in [12], but they are equivalent with the previous one by means of a simple transformation.

The next attempt is to split eq. (7) in a solvable system of nonlinear equations. Like in the linearization of the type I, we have the possibilities:

- *Linearization IIa*

$$\begin{aligned} F_{z+} &= F^{\alpha}, \forall \alpha, \\ F^{\alpha-1}(F_t + F_z) &= F_{z+,z-}. \end{aligned} \tag{9}$$

The advantage of this splitting is that the first equation of the system (9) is a Bernoulli one with the solution

$$F(z_+, z_-; t) = \{(1-\alpha)[z_+ + h(z_-, t)]\}^{\frac{1}{1-\alpha}}$$

where $h(z_-, t)$ is an arbitrary function. Introducing this expression in the second equation of the system one finds a linear equation for $h(z_-, t)$:

$$h_t + (1-\alpha)h_z = 0.$$

In this way, the general solution for the system (IV) is

$$F(z_+, z_-; t) = \left\{(1-\alpha)\left[z_+ + Cz_+\left(t - \frac{z_-}{1-\alpha}\right)\right]\right\}^{\frac{1}{1-\alpha}}$$

which gives

$$v(z_+, z_-, t) = \frac{1 + C\left(t - \frac{z}{1-\alpha}\right)}{z_+ + Cz + \left(t - \frac{z}{1-\alpha}\right)} = \frac{1}{z_+}. \tag{10}$$

Accordingly this nonlinear splitting gives a stationary solution, i.e. independent of the parameter of deformation t.

- *Linearization IIb*

$$\begin{aligned} F_{z+,z-} &= F^{\alpha} F_{z+} \\ F_t + F_z &= F^{\alpha+1}. \end{aligned} \tag{11}$$

From the first equation of the system we get

$$F_z = \frac{1}{\alpha+1} F^{\alpha+1} + \beta(z_-, t)$$

with β an arbitrary function. Introducing in the second equation we get

$$F_t + F_z = (\alpha+1)(F_z - \beta)$$

having the solution

$$F(z_+, z_-; t) = -\frac{\alpha+1}{\alpha}\int^z \beta(\xi, z_- + \alpha t - \xi)d\xi + h(z_+, z_- + \alpha t)$$

Now solving the first equation for h in the case of a general function β is a difficult task. In any case, for $\beta = 0$ the system can be solved having the solution

$$v(z_+, z_-; t) = \frac{f(z_+)}{f(z_+) - \alpha(1+\alpha)z_- + \alpha t}. \tag{12}$$

We remark that the nonlinear splitting of the bilinear form gives solutions which are less general than (8). A different approach to eq. (7) can be done using a special combination of the variables z_+ and z_- as a new independent variable. For example the well known solutions of cigar-type, or rational type, discussed in [10] can be obtained by introducing the following substitution:

$$z_{+} + z_{-} = \xi. \tag{13}$$

Then (2) becomes

$$v_t = (\ln v)_{\xi\xi}. \tag{14}$$

which has been extensively studied in [13].

Another simple symmetric combination of variables z_{+} and z_{-} is

$$z_{+} z_{-} = \xi. \tag{15}$$

Again, using the same machinery we end up with the following bilinear equation:

$$F_{\xi}(F_t + \xi F_{\xi}) = \xi F F_{\xi\xi} \tag{16}$$

and, of course

$$v(z_{+}, z_{-}; t) = \frac{\partial}{\partial \xi} \ln F .$$

As in the linearization I, the following alternatives are obvious:

- *Linearization IIIa*

Now, we can split (13) in the same way as before:

$$\begin{aligned} F_t + \xi F_{\xi} &= 0 \\ F_{\xi\xi} &= 0 \end{aligned}$$

with the general solution

$$F(\xi, t) = a\xi e^{-t} + b$$

where a,b are constants. In this case

$$v(z_{+}, z_{-}; t) = \frac{1}{z_{+} z_{-} + \frac{b}{a} e^{t}}. \tag{17}$$

- *Linearization IIIb*

$$\begin{aligned} F_{\xi} &= \xi F_{\xi\xi} \\ F_t + \xi F_{\xi} &= F. \end{aligned}$$

From the first equation of this system we have

$$F(\xi, t) = \frac{1}{2}\xi^2 h(t) + \mu(t)$$

and from the second $h(t) = ae^{-t}, \mu(t) = ce^{t}$ with a,c constants. Accordingly

$$v(z_+, z_-; t) = \frac{z_+ z_-}{\frac{1}{2}(z_+ z_-)^2 + \frac{c}{a} e^{2t}} \tag{18}$$

Other two splittings are presented in [12], but they give trivial or stationary solutions.

Finally, let us consider the following splittings of eq. (16) in solvable systems of nonlinear equations:

- *Linearization IV*

$$F_\xi = F^\alpha$$
$$F^{\alpha-1}(F_t + \xi F_\xi) = \xi F_{\xi\xi}.$$

From the first equation of the system we get

$$F = [(1-a)(\xi + g(t))]^{\frac{1}{1-a}},$$

but if we introduce it in the second equation we have $g'+1 = \alpha z$ which gives $g' = -1$ and $\alpha = 0$ and consequently F is trivial.

Other similar linearizations are presented in [12], but again they do not direct to new interesting solutions.

In what follows, let us summarize the obtained results and comment their physical relevance. The multitude of explicit solutions of the Ricci flow equation (3) proves that the linearization approach represents an efficient procedure to generate explicit solutions. Among them, the solution (8) is the most general and corresponds to the largest linearizable sector. As we mentioned above, in the form of solution (8), the role of the variables z_+ and z_- can be interchanged, taking into account that eq. (3) is symmetric in these variables. The exhaustive study of the Lie symmetries and similarity reductions performed on the quadratic equation (7) will be the subject of forthcoming work [14].

In the present paper, to illustrate the importance of the symmetries of eq. (7), we limit ourselves to solutions which depend on simple symmetrical combinations of the variables z_+ and z_-. First of all, we considered solutions which depend only on $\xi = z_+ + z_-$ and t. In this case we get eq. (14) which emerges in plasma physics and in the central limit approximation to Carleman's model of the Boltzmann equation [15]. In [13] it was pointed out that this equation has an *addition property*. To exhibit briefly this remarkable property, let us introduce $u = y_\xi$ and put eq. (14) in the equivalent form:

$$y_\xi y_t = y_{\xi\xi} + C y_\xi, \tag{19}$$

with C a constant. If $Y(\xi+\lambda t)$ and $Z(\xi-\lambda t)$ are particular solution of (19), then for every λ,

$$Y(\xi+\lambda t)+Z(\xi-\lambda t)$$

is a solution as well. It can consider that the nonlinearity in eq. (19) annihilates the nonlinear interaction.

Finally, let us remark that in all the above searches for solutions we did not assume a particular dependence on the parameter *t*. However, looking for a special dependence on *t*, we are able to produce new solutions of special interest. To exemplify this possibility, let us assume for $v(z_+,z_-,t)$ a linear dependence on *t*:

$$v(z_+,z_-,t)=tG(z_+,z_-)$$

and investigate the consequences of this ansatz for problems with axial symmetry. Therefore, assuming that the dependence on the variables z_+ and z_- is through the symmetrical combination $\xi=z_+z_-$, the Ricci flow equation (3) becomes

$$G=(\ln G)_\xi+\xi(\ln G)_{\xi\xi}$$

Making the substitution $G=\phi_\xi$ we end up with the equation

$$\phi\phi_\xi=\xi\phi_{\xi\xi}$$

Integrating this equation once with respect to ξ, we get the ordinary differential equation

$$\xi\phi_\xi=\frac{\phi^2}{2}+\phi+c. \tag{20}$$

where c is a constant. The solution of this equation is

$$\phi=-1+\sqrt{2c-1}\tan\left\{\frac{1}{2}\left[2\sqrt{2c-1}(b+\ln\xi)\right]\right\},$$

which implies that

$$v(z_+,z_-,t)=\frac{td^2}{2z_+z_-\cos^2\ln\left[b(z_+z_-)^{\frac{d}{2}}\right]} \tag{21}$$

where we denoted by $d=\sqrt{2c-1}$.

For $c=0$, solution (21) acquires a simpler form:

$$\phi=\frac{2a\xi}{a\xi-1},$$

where a is a constant. In this case we get for $v(z_+,z_-,t)$ the final form

$$v(z_+, z_-, t) = t\frac{2a}{(az_+ z_- - 1^2)} \tag{22}$$

We note that in physics we come often across such kind of solutions with a linear dependence on t . For example, we mention the renormalization group flow equation for a sigma model onto a purely gravitational target space. More precisely, the class of solutions (22) appears in the exact solutions of the renormalization group equations that describe the decay of conical singularities in non-compact spaces [10, 16].

3. Concluding remarks

It is a hopeless task to find the general solution of the Ricci flow equation in arbitrary number of dimensions. Fortunately, several important results depend only on the qualitative properties of the flows and not on exact solutions. Many times it is also useful to have explicit expressions or computer simulations. All these considerations determine us to continue the investigation of the Ricci flow equation in space-times with low dimensions.

The linearization approach represents a powerful procedure to generate explicit solutions of the Ricci flow equation. Adding supplementary assumptions connected with the symmetries of the concrete physical systems or assuming special dependence on parameter t it is possible to get effectively large class of solutions.

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